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$SU(3)$ Richardson–Gaudin models: three-level systems

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Abstract
We present the exact solution of the Richardson–Gaudin models associated with the $SU(3)$ Lie algebra. The derivation is based on a Gaudin algebra valid for any simple Lie algebra in the rational, trigonometric and hyperbolic cases. For the rational case additional cubic integrals of motion are obtained, whose number is added to that of the quadratic ones to match, as required from the integrability condition, the number of quantum degrees of freedom of the model. We discuss different $SU(3)$ physical representations and elucidate the meaning of the parameters entering in the formalism. By considering a bosonic mapping limit of one of the $SU(3)$ copies, we derive new integrable models for three level systems interacting with two bosons. These models include a generalized Tavis–Cummings model for three level atoms interacting with two modes of the quantized electric field.

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1. Introduction

The Richardson–Gaudin (RG) exactly solvable models [1, 2] can be traced back to the exact solution of the BCS Hamiltonian given by Richardson in the early 1960s [3] and to the integrable spin model developed by Gaudin in the seventies [4]. They are diagonalizable by Bethe ansatz techniques and are the so-called classical limit of two-dimensional vertex models. We refer the reader to [5], where the connection between the RG models and the inhomogeneous XXX vertex model with twisted boundary conditions is established in detail for the case of the rank-1 $SU(2)$ algebra. Previously, in [6] it was shown that the solution of the Gaudin models (those without linear term in the integrals of motion; cf (19)) associated with more general Lie algebras, can be used to get a solution of the corresponding classical Yang–Baxter equations (CYBE). This connection has been largely exploited to diagonalize the RG models (see [2, 7] for some examples). In practice this method consists of using the already known solutions of the general-Lie-algebra Yang–Baxter equations (obtained by using the quantum inverse scattering method) to obtain the corresponding solution of the RG

models by taking the classical limit of the respective Bethe equations. From a conformal field theory context, Asorey, Falceto and Sierra [8] obtained the integrals of motion and respective eigenvalues of the rational RG models for any simple Lie algebra as a limiting case of the Chern–Simons theory.

Here we follow an alternative and more direct approach to diagonalize the RG models. Though essentially equivalent, it differs in practice. This approach, where no reference to the YBE is necessary, is similar to the presented by Ushveridze [9] for the rational case. The method is based on the introduction of an infinite-dimensional algebra (the Gaudin algebra) associated with the Lie-algebra of a simple group. By taking a Casimir-like operator in the Gaudin algebra, one gets the transfer matrix of the YBE approach and, consequently, a set of independent quadratic integrals of motion. In [10] this formalism was extended to the trigonometric and hyperbolic cases for the rank-1 algebras $SU(2)$ and $SU(1,1)$. In this contribution we extend the Gaudin algebra to the trigonometric and hyperbolic cases for any simple Lie algebra.

As an application of the method, we work out in detail the RG models associated with the rank-2 $SU(3)$ algebra. Most of the physical applications of the RG models presented so far are based on the rank-1 algebras $SU(2)$ or $SU(1,1)$. They cover a wide variety of physical problems ranging from pairing Hamiltonains for fermion or boson systems [11, 12], to spin models or generalized Tavis–Cumming models [13]. Recently, some applications to physical problems of RG models based on higher rank algebras have been published. The detailed derivation of the exact solution was considered for the $SO(5)$ algebra of proton–neutron pairing in [14] and with more generality including numerical applications in [15]. In [16] the RG model based on the non-compact $SO(3,2)$ algebra is used in the context of the Interacting Boson Model II, which describes the interaction between two different species of bosons (π and ν) for each non-degenerated level. In [7] the $SU(4)$ RG models are used to describe the interaction between two superconducting systems. Here we discuss different physical scenarios of potential interest, associated with the $SU(3)$ RG models. They include dipole–dipole interactions between three level atoms, isospinorial pairing and generalized Tavis–Cummings models of three level atoms interacting with two different bosonic modes [17].

Contrary to rank-1 RG models, in higher rank algebras the number of independent integrals of motion is not exhausted by those obtained from the quadratic Casimir-like operators. More integrals of motion can be obtained by considering higher degree operators. Unfortunately, as was shown in [18], the Casimir-like analogy to obtain integrals of motions is valid up to degree-three Casimir operators. More general formulae are needed to get integrals of motion of degree greater than three [19]. Nevertheless, since the $SU(3)$ algebra has two independent Casimir operators of degree two and three respectively, the Casimir-like analogy can be used to obtain the complet set of integrals of motion. In this contribution it is verified that, for the rational $SU(3)$ RG models, the number of independent integrals of motion coincides with the number of quantum degrees of freedom as defined in [20]. For other higher-rank algebras, which have at least quartic Casimir operators, the more general formulae of [19] are needed to obtain the complete set of integrals of motion.

This paper is organized as follows. In section 2, the general Gaudin algebra and quadratric Casimir-like operators are introduced. A particular realization of the Gaudin algebra in terms of a direct product of L Lie algebra copies and complex valuated functions is introduced. The conditions to be satisfied by these functions are established, which are a generalization of the Gaudin conditions of the $SU(2)$ case. Three particular solutions to the Gaudin conditions are found (rational, trigonometric and hyperbolic) and it is shown how to obtain the corresponding sets of quadratic integrals of motion from the Casimir-like Gaudin operator. Special interest

is paid to the introduction of a linear term in the integrals of motion through the addition of a constant shift to the Cartan members of the Gaudin algebra. Explicit formulae for the quadratic integrals of motion and their eigenvalues are given. In section 3, focusing on the $SU(3)$ algebra, we apply the formalism of section 2 to obtain closed expressions for the quadratic integrals of motion and their eigenvalues in the more general scenario of arbitrary $SU(3)$ irreducible representations. As discussed above, in order to satisfy the condition of integrability more integrals of motion are needed, which must be at least cubic in the generators. For the rational version we present new sets of integrals of motion coming from the Casimir-like Gaudin operator of degree three, and it is verified that the total number of quantum constants of motion coincides with the number of quantum degrees of freedom. In section 4, the physical meaning of the variables appearing in the integrals of motion and eigenvalues is discussed for different $SU(3)$ physical realizations. Additionally, the limit of infinite degeneracy for a copy of the $SU(3)$ algebra is used to get a family of integrals of motion related to a generalized Tavis–Cummings model for three level atoms interacting with two species of bosonic excitations. Conclusion are given in the last section.

2. Richardson–Gaudin models from a generalized Gaudin algebra

Let us begin by considering a simple Lie algebra, expressed in terms of its Cartan–Weyl decomposition [21, 22]:

$$[\mathbf{S}^a, \mathbf{S}^b] = \sum_c C_c^{ab} \mathbf{S}^c, \tag{1}$$

where the nonzero structure constants are given by

$$C_\alpha^{i\alpha} = -C_\alpha^{\alpha i} = \alpha^i, \qquad C_i^{\alpha-\alpha} = \frac{2}{|\alpha|^2}\alpha^i, \qquad C_{\alpha+\beta}^{\alpha\beta} = N_{\alpha,\beta},$$

the Latin index runs over the r Cartan–Weyl members of the Cartan subalgebra (r = rank of the group), and Greek indexes refer to the roots. α^i denotes the component i of the root α. Associated with this algebra we propose the following infinite-dimensional Gaudin algebra:

$$[\mathbf{S}^a(\lambda), \mathbf{S}^b(\mu)] = \sum_c C_c^{ab}\{X_b(\lambda-\mu)\mathbf{S}^c(\lambda) - X_a^*(\lambda-\mu)\mathbf{S}^c(\mu)\}, \tag{2}$$

where $X_a(\lambda)$ are meromorphic functions associated with the member a of the Gaudin algebra and λ is a complex variable. From these commutation rules, it can be proved that the Casimir-like operators of the Gaudin algebra,

$$\mathbf{K}(\lambda) \equiv \sum_{ab} g_{ab}\mathbf{S}^a(\lambda)\mathbf{S}^b(\lambda), \tag{3}$$

where g_{ab} is the inverse of the Killing form, commute among themselves:

$$[\mathbf{K}(\lambda), \mathbf{K}(\mu)] = 0 \quad \forall\lambda, \mu. \tag{4}$$

A realization of the Gaudin algebra is given by

$$\mathbf{S}^a(\lambda) = \sum_{m=1}^{L} \mathbf{S}_m^a X_a(z_m - \lambda), \tag{5}$$

where the index m runs over L different copies of the Lie algebra, $\mathbf{S}_m^a$ is a generator of the mth copy, and z_m is a set of completely free real parameters. The operators $\mathbf{S}^a(\lambda)$ act upon the space $V_1 \otimes \cdots \otimes V_L$, with V_m an irreducible representation (irrep) of the Lie algebra. Given this specific realization, the commutation rules (2) impose some conditions on the functions $X_a(\lambda)$. All the functions associated with the elements of the Cartan subalgebra are equal, real

valuated and antisymmetric:

$$X_i(\lambda) \equiv Z(\lambda) = Z^*(\lambda) \qquad \forall i \tag{6}$$

$$Z(\lambda) = -Z(-\lambda). \tag{7}$$

The reality condition on the functions $Z(\lambda)$ comes from keeping the hermiticity of the Cartan-subalgebra in its Gaudin counterpart $[\mathbf{S}^i(\lambda) = (\mathbf{S}^i(\lambda))^\dagger]$. On the other hand, since $\mathbf{S}^\alpha(\lambda) = (\mathbf{S}^{-\alpha}(\lambda))^\dagger$ the functions associated with the Gaudin ladder operators must satisfy

$$X_\alpha(\lambda) = X^*_{-\alpha}(\lambda) \qquad \forall \alpha. \tag{8}$$

Additionally they have to be anti-hermitic:

$$X_\alpha(\lambda) = -X^*_\alpha(-\lambda). \tag{9}$$

The last conditions that have to be satisfied by the functions $Z(\lambda)$ and $X_\alpha(\lambda)$ are a generalized version of the $SU(2)$-Gaudin conditions:

$$\begin{aligned} &X_\alpha(\mu-\lambda)X_\alpha(\lambda) + Z(\lambda-\mu)X_\alpha(\mu) - Z(\lambda)X_\alpha(\mu) = 0 \quad \forall\alpha \\ &X_\beta(\mu-\lambda)X_{\alpha+\beta}(\lambda) + X_\alpha(\lambda-\mu)X_{\alpha+\beta}(\mu) - X_\alpha(\lambda)X_\beta(\mu) = 0 \quad \forall\alpha, \quad \beta \text{ such that} \\ &\qquad \alpha+\beta \text{ is a root.} \end{aligned} \tag{10}$$

Using the structure constants, equations (2), (6) and (8), and denoting the Gaudin-operators by the more common notation $\mathbf{S}^i(\lambda) = \mathbf{H}^i(\lambda)$ and $\mathbf{S}^\alpha(\lambda) = \mathbf{E}^\alpha(\lambda)$, the following form for the Gaudin algebra is obtained:

$$[\mathbf{H}^i(\lambda), \mathbf{E}^\alpha(\mu)] = \alpha^i\{X_\alpha(\lambda-\mu)\mathbf{E}^\alpha(\lambda) - Z(\lambda-\mu)\mathbf{E}^\alpha(\mu)\} \tag{11}$$

$$[\mathbf{E}^\alpha(\lambda), \mathbf{E}^{-\alpha}(\mu)] = \frac{2}{|\alpha|^2}X^*_\alpha(\lambda-\mu)\{\alpha\cdot\mathbf{H}(\lambda) - \alpha\cdot\mathbf{H}(\mu)\} \tag{12}$$

$$[\mathbf{E}^\alpha(\lambda), \mathbf{E}^\beta(\mu)] = N_{\alpha,\beta}\{X_\beta(\lambda-\mu)\mathbf{E}^{\alpha+\beta}(\lambda) - X^*_\alpha(\lambda-\mu)\mathbf{E}^{\alpha+\beta}(\mu)\}, \tag{13}$$

where $\alpha\cdot\mathbf{H} = \sum_i^r \alpha^i\mathbf{H}^i$. From equation (12) it is clear that the Cartan subalgebra members of the Gaudin algebra are defined up to a constant term, which can be freely added without altering the commutation relationships:

$$\mathbf{H}^i(\lambda) = \sum_m \mathbf{H}^i_m Z(z_m-\lambda) + k^i\mathbf{1}, \tag{14}$$

where $\mathbf{1}$ is the unity operator in $V_1\otimes\cdots\otimes V_L$, and k^i are r free real parameters.

In the rest of the paper, we will only consider the case where all the functions associated with the positive root operators are equal ($X_\alpha(\lambda) = X(\lambda), \forall\alpha > 0$) and let for the future a more general discussion. In this case, three solutions to the Gaudin conditions (10) are as follows.

- The rational solution

$$X(z_n-z_m) = Z(z_n-z_m) = \frac{1}{z_n-z_m}. \tag{15}$$

- The trigonometric solution

$$X(z_n-z_m) = \frac{\exp[\mathbf{i}(z_n-z_m)]}{\sin(z_n-z_m)}, \qquad Z(z_n-z_m) = \cot(z_n-z_m). \tag{16}$$

- The hyperbolic solution

$$X(z_n - z_m) = \frac{\exp[\mathbf{i}(z_n - z_m)]}{\sinh(z_n - z_m)}, \qquad Z(z_n - z_m) = \coth(z_n - z_m). \tag{17}$$

The Casimir-like operator defined in (3) acts as a generator of the quadratic RG integrals of motion. From (10) and their solutions (15), (16) and (17), it is easy to show that the operator $\mathbf{K}(\lambda)$ can be written as

$$\mathbf{K}(\lambda) = \sum_{m=1}^{L} \frac{\mathbf{C}_m^{(2)}}{(\lambda - z_m)^2} - 2\sum_{m=1}^{L} \frac{\mathbf{R}_m}{\lambda - z_m} + \cdots, \tag{18}$$

where $\mathbf{C}_m^{(2)}$ is the Casimir of degree two of the mth Lie algebra copy. The operators $\mathbf{R}_m$ read

$$\mathbf{R}_m = \boldsymbol{\xi}_m + \sum_{n(n\neq m)}^{L} \left(\sum_{\alpha>0} \frac{|\alpha|^2}{2} \left(X(z_n - z_m)\mathbf{E}_m^{\alpha}\mathbf{E}_n^{-\alpha} + X^*(z_n - z_m)\mathbf{E}_m^{-\alpha}\mathbf{E}_n^{\alpha} \right) + Z(z_n - z_m) \sum_{i=1}^{r} \mathbf{H}_m^i \mathbf{H}_n^i \right), \tag{19}$$

where the linear term $\boldsymbol{\xi}_m = \sum_{i=1}^{r} k^i \mathbf{H}_m^i$ is a member of the m th Cartan subalgebra coming from the constant term shift introduced in (14). Note that the coefficients k^i do not have an index m, i.e., even if each $\boldsymbol{\xi}_m$ belongs to a different copy of the Lie algebra, the same linear combination with respect to the corresponding basis of the Cartan subalgebra is assumed for all the copies.

The commutativity of the Casimir-like Gaudin operators (4) implies commutativity among the set of operators $\mathbf{R}_m$:

$$[\mathbf{R}_m, \mathbf{R}_n] = 0 \qquad \forall n, m = 1, \ldots, L. \tag{20}$$

The eigenfunctions and eigenvalues of the set of integrals of motion $\mathbf{R}_m$ are obtained by considering a Bethe ansatz, which is solely written in terms of the lowering operators of the Gaudin algebra and a set of parameters to be determined. This calculation is performed in [9] for the rational case. Repeating the same steps for the trigonometric and hyperbolic cases is a straightforward but laborious task. It consists of applying the Casimir-like Gaudin operator to the ansatz. This application yields one term proportional to the original ansatz and terms which are not proportional. From the former one, one can obtain the eigenvalues of $\mathbf{R}_m$ by performing a similar expansion as in (18). By imposing the annulation of all the non-proportional terms, one gets the equations that determine the parameters entering in the ansatz. We present the results. The eigenvalues of the operators $\mathbf{R}_m$ are given by

$$r_m = \sum_{ab}^{r} \xi^a F_{ab} \Lambda_m^b + \sum_{n(n\neq m)}^{L} Z(z_n - z_m)\Lambda_m \cdot F \cdot \Lambda_n + \sum_{a=1}^{r}\sum_{k=1}^{M_a} \Lambda_m^a \frac{|\alpha_a|^2}{2} Z\big(z_m - E_k^a\big), \tag{21}$$

where α_a are the simple roots of the algebra (there are as many simple roots as the rank of the algebra). The r coefficients ξ^a are related to the parameters k^i of the linear term through:

$$\boldsymbol{\xi}_m = \sum_i k^i \mathbf{H}_m^i = \sum_{ab} \xi^a F_{ab} \mathbf{h}_m^b, \tag{22}$$

where $\mathbf{h}_m^b$ are the members of the Chevalley basis of the mth Cartan subalgebra. The Chevalley basis is defined as

$$\mathbf{h}_m^a = \frac{2}{|\alpha_a|^2}(\alpha_a \cdot \mathbf{H}_m),$$

where α_a is one of the r simple roots and $\mathbf{H}_m$ is the already introduced Cartan–Weyl basis of the Cartan subalgebra. F_{ab} is a $r \times r$ matrix called the quadratic form of the algebra, which is related to the Cartan matrix through $F_{ab} = (A^{-1})_{ab}\frac{|\alpha_b|^2}{2}$. The Cartan matrix codifies completely the structure of the algebra and is defined in terms of the simple roots: $A_{ab} = \frac{2}{|\alpha_a|^2}(\alpha_a, \alpha_b)$, where the scalar product of the roots is defined as $(\alpha_a, \alpha_b) = \sum_i \alpha_a^i \alpha_b^i$. Λ_m^b are the Dynkin labels (the eigenvalues of the Chevalley basis) in the highest weight state of the mth irreducible representation V_m, and the product $\Lambda_m \cdot F \cdot \Lambda_n$ is equal to $\sum_{ab} \Lambda_m^a F_{ab} \Lambda_n^b$. The variables E_k^a entering in (21) determine the common eigenfunctions of the operators $\mathbf{R}_m$, and are the solutions of the following Richardson–Bethe equations:

$$\sum_{b=1}^{r}\sum_{k'=1}^{M_b} A_{ba} Z\left(E_{k'}^b - E_k^a\right) - \sum_{m=1}^{L} \Lambda_m^a Z\left(z_m - E_k^a\right) = \xi^a, \qquad (k = 1, \ldots, M_a). \tag{23}$$

The number of these parameters E_k^a is

$$M_a = \frac{2}{|\alpha_a|^2}\sum_{b=1}^{r} F_{ba}\left(\sum_m \Lambda_m^b - \lambda_0^b\right), \tag{24}$$

where λ_0^b are the eigenvalues of the overall operators $\sum_m^L \mathbf{h}_m^b$ (the sum of the Chevalley basis over all the copies). These overall operators commute with the integrals of motion $\mathbf{R}_m$ (see (35)), therefore their eigenvalues $\left(\lambda_0^b\right)$ are conserved quantities for any Hamiltonian defined as a function of the integrals of motion $\mathbf{R}_m$. Equations (19), (21) and (23) extend the results presented in [8, 9] for the rational case, to the trigonometric and hyperbolic ones.

3. The $SU(3)$ algebra

3.1. Quadratic integrals of motion

In this section, we will apply the formulae of the previous one in the particular case of the rank-2 algebra $SU(3)$. We consider L copies of a $U(3)$ algebra:

$$[\mathbf{K}_{\alpha\beta m}, \mathbf{K}_{\alpha'\beta' m}] = \delta_{\beta\alpha'}\mathbf{K}_{\alpha\beta' m} - \delta_{\alpha\beta'}\mathbf{K}_{\alpha'\beta m}, \tag{25}$$

where the index m refers to the mth copy of the Lie algebra, and $\alpha, \beta = 1, 2, 3$. From these commutations rules it is easy to prove that $[\mathbf{K}_{11m} + \mathbf{K}_{22m} + \mathbf{K}_{33m}, \mathbf{K}_{\alpha\beta m}] = 0\ \forall\alpha, \beta$. Therefore this sum must be proportional to the operator unity in the $SU(3)$-irrep V_m and, consequently, an integral of motion:

$$\mathbf{K}_{11m} + \mathbf{K}_{22m} + \mathbf{K}_{33m} \equiv \mathbf{n}_m = n_m \mathbf{1}. \tag{26}$$

This condition reduces the number of independent generators from 9 to 8, the dimension of the $SU(3)$-algebra. A Cartan decomposition of the $SU(3)$ algebra is as follows.

- A maximal Abelian subalgebra of Hermitian operators (the Cartan subalgebra $\mathcal{H}$) is provided by the operators $\mathbf{K}_{\alpha\alpha m}$. However, due to the condition (26) only two of them are independent. Two different Cartan subalgebra bases adequate for our purposes are the Cartan–Weyl basis (we are using the normalization convention of [21]):

$$\mathbf{H}_m^1 = \frac{\mathbf{K}_{11m} - \mathbf{K}_{33m}}{\sqrt{2}}, \qquad \mathbf{H}_m^2 = \frac{2\mathbf{K}_{22m} - \mathbf{K}_{33m} - \mathbf{K}_{11m}}{\sqrt{6}}, \tag{27}$$

  and the Chevalley basis

$$\mathbf{h}_m^1 = \mathbf{K}_{11m} - \mathbf{K}_{22m}, \qquad \mathbf{h}_m^2 = \mathbf{K}_{22m} - \mathbf{K}_{33m}. \tag{28}$$

  These bases generate the Cartan subalgebra $\mathcal{H}_m = \mathrm{span}\left(\mathbf{H}_m^1, \mathbf{H}_m^2\right) = \mathrm{span}\left(\mathbf{h}_m^1, \mathbf{h}_m^2\right)$.

- The positive root vector space is spanned by the raising operators:

$$\mathcal{E}_m^+ = \mathrm{span}(\mathbf{K}_{\alpha\beta m}) \qquad \text{with} \quad \beta > \alpha. \tag{29}$$

- The lowering operators are the Hermitian conjugated of the previous ones. The negative root vector space is

$$\mathcal{E}_m^- = \mathrm{span}(\mathbf{K}_{\alpha\beta m}) \qquad \text{with} \quad \beta < \alpha. \tag{30}$$

The mth Lie algebra of $SU(3)$ is given by the direct sum $\mathcal{L}_{SU(3)m} = \mathcal{E}_m^- \bigoplus \mathcal{H}_m \bigoplus \mathcal{E}_m^+$. The roots of the algebra can be obtained from the commutation relations between the Cartan–Weyl basis and the root vectors:

$$\left[\mathbf{H}_m^i, \mathbf{K}_{\alpha\beta m}\right] = \alpha_{\alpha\beta}^i \mathbf{K}_{\alpha\beta m} \qquad \alpha \neq \beta. \tag{31}$$

The algebra $SU(3)$ has three positive roots ($\beta > \alpha$), they are

$$\begin{aligned} \alpha_{12} &= \left(\alpha_{12}^1, \alpha_{12}^2\right) = \tfrac{1}{\sqrt{2}}(1, -\sqrt{3}) \\ \alpha_{23} &= \left(\alpha_{23}^1, \alpha_{23}^2\right) = \tfrac{1}{\sqrt{2}}(1, \sqrt{3}) \\ \alpha_{13} &= \left(\alpha_{13}^1, \alpha_{13}^2\right) = (\sqrt{2}, 0). \end{aligned} \tag{32}$$

It is easy to see that the simple roots are α_{12} and α_{23}, from which all the other roots can be obtained by a linear combination of integer coefficients. The non-simple positive root is $\alpha_{13} = \alpha_{12} + \alpha_{23}$, whereas the negative roots are $\alpha_{21} = -\alpha_{12}, \alpha_{32} = -\alpha_{23}$, and $\alpha_{31} = -\alpha_{12} - \alpha_{23}$. All the roots have square norm equal to 2: $|\alpha_{\alpha\beta}|^2 = 2$. The Cartan matrix is $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. From this expression we obtain the quadratic form $F = \frac{1}{3}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

By using all these results, we can write the integrals of motion (19) for the $SU(3)$ case. The rational integrals of motion read

$$\mathbf{R}_m = -\xi^1 \mathbf{K}_{22m} - (\xi^1 + \xi^2)\mathbf{K}_{33m} + C_m \mathbf{1} + \sum_{n(n\neq m)}^{L} \frac{\sum_\beta \sum_\alpha \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha n}}{z_n - z_m}. \tag{33}$$

For the trigonometric and hyperbolic cases ($\mathrm{ct}(z_m - z_n) = \cot(z_m - z_n)$ and $\mathrm{ct}(z_m - z_n) = \coth(z_m - z_n)$ respectively), the integrals of motion are

$$\begin{aligned} \mathbf{R}_m = -\xi^1 \mathbf{K}_{22m} - (\xi^1 + \xi^2)\mathbf{K}_{33m} + C_m \mathbf{1} + \sum_{n(n\neq m)}^{L} \Bigg[\mathrm{ct}(z_n - z_m) \sum_\beta \sum_\alpha \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha n} \\ + \mathbf{i} \sum_{\beta>\alpha} (\mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha n} - \mathbf{K}_{\beta\alpha m}\mathbf{K}_{\alpha\beta n}) \Bigg]. \end{aligned} \tag{34}$$

The constant term appearing in (33) and (34) is $C_m = -(n_m/3)\sum_{l(l\neq m)} Z(z_l - z_m) n_l + \frac{(2\xi^1+\xi^2)}{3} n_m$.

3.2. More integrals of motion

By direct calculation, it can be shown that the overall operators

$$\mathbf{N}_\alpha = \sum_{m=1}^{L} \mathbf{K}_{\alpha\alpha m}, \qquad \text{with} \quad \alpha = 1, 2, 3, \tag{35}$$

commute (in the trigonometric, hyperbolic, and rational cases) with the integrals $\mathbf{R}_m$. One of these operators must be independent of the already introduced integrals of motion. From

(26) it is clear that $\mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3 = \sum_m \mathbf{n}_m$, and from the integrals of motion (33) or (34) it is derived that $\sum_m (C_m \mathbf{1} - \mathbf{R}_m) = \xi^1 \mathbf{N}_2 + (\xi^1 + \xi^2)\mathbf{N}_3$. This fact implies that the integrals $\mathbf{R}_m$ do not exhaust all the possible integrals of motion. In [18] it was shown for the rational case that the Casimir-like Gaudin operators of degree three commute among them and with the quadratic Casimir-like operators (3). The $SU(3)$ Casimir operator of degree three is $\mathbf{C}^{(3)} = \sum_{\alpha\beta\gamma} \mathbf{K}_{\alpha\beta}\mathbf{K}_{\beta\gamma}\mathbf{K}_{\gamma\alpha}$, from here it is clear that the Gaudin counterpart must read $\mathbf{K}^{(3)}(\lambda) = \sum_{\alpha\beta\gamma} \mathbf{K}_{\alpha\beta}(\lambda)\mathbf{K}_{\beta\gamma}(\lambda)\mathbf{K}_{\gamma\alpha}(\lambda)$, with $\mathbf{K}_{\alpha\beta}(\lambda)$ a generator of the $SU(3)$-Gaudin algebra. This family of operators satisfies, at least for the rational case, $[\mathbf{K}^{(3)}(\lambda), \mathbf{K}^{(3)}(\mu)] = 0$ and $[\mathbf{K}^{(3)}(\lambda), \mathbf{K}(\mu)] = 0, \forall\lambda, \mu$. By expanding $\mathbf{K}^{(3)}(\lambda)$, as we did for $\mathbf{K}(\lambda)$ in (18), we get

$$\mathbf{K}^{(3)}(\lambda) = -\sum_{m=1}^{L} \frac{\mathbf{C}_m^{(3)}}{(\lambda - z_m)^3} + 3\sum_{m=1}^{L} \frac{\mathbf{P}_{3m}}{(\lambda - z_m)^2} - 3\sum_{m=1}^{L} \frac{\mathbf{R}_{3m}}{\lambda - z_m} + \sum_{\alpha=1}^{3} A_\alpha \mathbf{1}, \tag{36}$$

where $\mathbf{C}_m^{(3)}$ is the cubic-Casimir of the mth Lie algebra copy, and the operators $\mathbf{P}_{3m}$ and $\mathbf{R}_{3m}$ are

$$\mathbf{P}_{3m} = \sum_{n(n\neq m)}^{L} \sum_{k\left(\substack{k\neq n\\ k\neq m}\right)}^{L} \frac{\sum_{\alpha\beta\gamma} \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\gamma n}\mathbf{K}_{\gamma\alpha k}}{(z_n - z_m)(z_k - z_m)} + \sum_{n(n\neq m)}^{L} \frac{\sum_{\alpha\beta\gamma} \mathbf{K}_{\alpha\beta n}\mathbf{K}_{\beta\gamma m}(\mathbf{K}_{\gamma\alpha n} - \mathbf{K}_{\gamma\alpha m})}{(z_m - z_n)^2}$$
$$+ \sum_{n(\neq m)} \frac{\sum_{\alpha\beta} (A_\alpha \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha n} + A_\alpha \mathbf{K}_{\alpha\beta n}\mathbf{K}_{\beta\alpha m})}{z_m - z_n} + \sum_{\alpha} A_\alpha A_\alpha \mathbf{K}_{\alpha\alpha m} \tag{37}$$

$$\mathbf{R}_{3m} = \sum_{n(n\neq m)}^{L} \frac{\sum_{\alpha\beta\gamma} \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\gamma m}\mathbf{K}_{\gamma\alpha n}}{z_m - z_n} + \sum_{\alpha\beta} A_\alpha \mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha m}. \tag{38}$$

The coefficients A_α are related to the parameters of the linear term in $\mathbf{R}_m$: $A_1 = (2\xi^1 + \xi^2)/3$, $A_2 = (\xi^2 - \xi^1)/3$ and $A_3 = -(\xi^1 + 2\xi^2)/3$. The operators $\mathbf{P}_{3m}$, $\mathbf{R}_{3m}$ and $\mathbf{R}_m$ form a complete set of mutually commuting operators. From the expression for the integrals $\mathbf{P}_{3m}$, it is easy to show that the three operators $\mathbf{N}_\alpha$ (35) can be expressed as a linear combination of the operators $\mathbf{R}_m$, $\mathbf{n}_m$ and $\mathbf{P}_{3m}$. The commutativity of the cubic Casimir-like Gaudin operators has been proved in [18] for the rational case, we think this result can be extended to the trigonometric and hyperbolic ones.

3.3. Highest weight states and quantum-dynamical degrees of freedom

The highest weight states (unique for any finite irrep of a simple algebra) are defined by the conditions:

$$\mathbf{K}_{\alpha\beta m}|\Lambda_m\rangle = 0 \qquad \text{for all} \quad \beta > \alpha. \tag{39}$$

The eigenvalues of the members of the Chevalley basis in these states are integer positive numbers and allow us to label the irrep,

$$\mathbf{h}_m^1|\Lambda_m\rangle = \Lambda_m^1|\Lambda_m\rangle = (k_{1m} - k_{2m})|\Lambda_m\rangle \qquad \mathbf{h}_m^2|\Lambda_m\rangle = \Lambda_m^2|\Lambda_m\rangle = (k_{2m} - k_{3m})|\Lambda_m\rangle, \tag{40}$$

where $k_{\alpha m}$ are the eigenvalues of the operators $\mathbf{K}_{\alpha\alpha m}$ in the highest weight state, and we have used (28). The numbers $[k_{1m}k_{2m}k_3]$ determine the Young diagram of the irrep V_m and satisfy $k_{1m} + k_{2m} + k_{3m} = n_m$. All the members of the irrep can be obtained by acting the lowering operators upon the previous highest weight state.

In general, once we have completely established for each m the values of the set $\left(\Lambda_m^1, \Lambda_m^2, n_m\right)$ (or equivalently $[k_{1m}k_{2m}k_{3m}]$), we need three extra numbers for each copy

of the Lie algebra to determine a complete basis of the quantum system. Consequently a complete basis for the Hilbert space $V_1 \otimes \cdots \otimes V_L$ is

$$|\Psi\rangle = \left| \begin{matrix} n_1 & \cdots & n_m & \cdots & n_L \\ (\Lambda_1^1, \Lambda_1^2) & \cdots & (\Lambda_m^1, \Lambda_m^2) & \cdots & (\Lambda_L^1, \Lambda_L^2) \\ t_{11}, t_{21}, t_{31} & \cdots & t_{1m}, t_{2m}, t_{3m} & \cdots & t_{1L}, t_{2L}, t_{3L} \end{matrix} \right\rangle_\rho \tag{41}$$

where ρ is a multiplicity number to take account of any other completely degenerated quantum number not considered explicitly here. The labels t_{1m}, t_{2m}, t_{3m} are the three necessary quantum numbers to characterize completely a state within an $SU(3)$ irrep. The number of non-completely degenerated quantum numbers necessary to determine unambiguously a basis' member is the number of quantum-dynamical degrees of freedom of the system [20]. From (41) it is deduced that, in the present case, the number of quantum degrees of freedom is

$$d = 3L. \tag{42}$$

To guarantee the integrability of the system we need $3L$ integrals of motion. L of them are provided by the operators $\mathbf{R}_m$, the other $2L$ integrals of motion are the polynomial operators of degree three ($\mathbf{R}_{3m}$ and $\mathbf{P}_{3m}$) introduced in the previous subsection.

3.4. Eigenvalues

We can explicitly write the eigenvalues of the operators (33) and (34), which are the $SU(3)$ version of the general formula (21):

$$\begin{aligned} r_m = -\xi^1 k_{2m} - (\xi^1 + \xi^2) k_{3m} + C_m + \sum_{n(n\neq m)}^{L} \left(Z(z_n - z_m) \sum_{\alpha=1}^{3} k_{\alpha m} k_{\alpha n} \right) \\ + (k_{1m} - k_{2m}) \sum_{k=1}^{M_1} Z(z_m - E_k) + (k_{2m} - k_{3m}) \sum_{l=1}^{M_2} Z(z_m - \omega_l), \end{aligned} \tag{43}$$

where C_m is the same constant found in the operators (33) and (34), and we have redefined the parameters $E_k \equiv E_k^1$ and $\omega_l \equiv E_l^2$. These parameters determine the eigenfunction of the operators $\mathbf{R}_m$, and are the solutions of the following Richardson–Bethe equations:

$$\sum_{k'(k'\neq k)}^{M_1} 2Z(E_{k'} - E_k) - \sum_{l=1}^{M_2} Z(\omega_l - E_k) - \sum_{m=1}^{L} (k_{1m} - k_{2m}) Z(z_m - E_k) = \xi^1 \qquad (k = 1, \ldots, M_1) \tag{44}$$

$$-\sum_{k=1}^{M_1} Z(E_k - \omega_l) + \sum_{l'(l'\neq l)}^{M_2} 2Z(\omega_{l'} - \omega_l) - \sum_{m=1}^{L} (k_{2m} - k_{3m}) Z(z_m - \omega_l) = \xi^2 \qquad (l = 1, \ldots, M_2). \tag{45}$$

To determine the number of parameters E_k and ω_l (M_1 and M_2 respectively), note that the members of the overall Chevalley basis can be expressed in terms of the integrals (35),

$$\sum_m \mathbf{h}_m^1 = \mathbf{N}_1 - \mathbf{N}_2, \qquad \sum_m \mathbf{h}_m^2 = \mathbf{N}_2 - \mathbf{N}_3,$$

consequently their eigenvalues are $\left(\lambda_0^1, \lambda_0^2\right) = (N_1 - N_2, N_2 - N_3)$. By using this result, the relation $\sum_m (k_{1m} + k_{2m} + k_{3m}) = N_1 + N_2 + N_3$, the quadratic form and the labels of the $SU(3)$ irreps (40) in the general formula (24), one gets

$$M_1 = \sum_m^L k_{1m} - N_1, \qquad M_2 = N_3 - \sum_m^L k_{3m}. \tag{46}$$

The values M_1 and M_2 allow us to label the different invariant subspaces of the Hilbert space. We denote these subspaces by $V(M_1, M_2) \subset V_1 \otimes \cdots \otimes V_L$. The different solutions of the Richardson–Bethe equations define a set of eigenfunctions which span entirely the subspace $V(M_1, M_2)$. By considering all the possible values of M_1 and M_2 and the set of complete solutions of the corresponding Richardson–Bethe equations, we get a basis for the entire Hilbert space $V_1 \otimes \cdots \otimes V_L = \bigoplus_{M_1, M_2} V(M_1, M_2)$.

For a given set of L $SU(3)$ irreps ($[k_{1m} k_{2m} k_{3m}]$ with $m = 1, \ldots, L$), the possible values of M_1 and M_2 are

$$0 \leqslant M_1 \leqslant \sum_m (k_{1m} - k_{3m})$$

$$\max\left(0, M_1 - \sum_m (k_{1m} - k_{2m})\right) \leqslant M_2 \leqslant \sum_m (k_{1m} - k_{3m}) + \min\left(0, M_1 - \sum_m (k_{1m} - k_{2m})\right).$$

4. Physical models related to the $SU(3)$ RG models

4.1. Particle–hole representation

An explicit realization of the U(3) algebra (25) can be obtained by considering Ω_m three-level atoms of type m:

$$\mathbf{K}_{\alpha\beta m} = \sum_{\mu=1}^{\Omega_m} \mathbf{c}^\dagger_{\alpha\mu m} \mathbf{c}_{\beta\mu m}, \tag{47}$$

where $\mathbf{c}^\dagger_{\alpha\mu m}$ and $\mathbf{c}_{\beta\mu m}$ are fermion creation and annihilation operators, respectively. The indexes α and β label each of the three levels, whereas the index μ runs over all the atoms of type m. We can introduce different types of atoms (let us say L different types), each type of atom associated with a copy of the $SU(3)$ algebra ($m = 1, \ldots, L$).

The operators $\mathbf{K}_{\alpha\beta m}$ have a simple physical interpretation. For $\alpha = \beta$ the operators are the number operators: $\mathbf{K}_{\alpha\alpha m} = \mathbf{n}_{\alpha m}$. The raising operators ($\alpha < \beta$) take a particle (excitation) from a level to a lower one, whereas the lowering operators ($\alpha > \beta$) take a particle from a level to a higher one. In this context, the condition (26) is simply the conservation of the number of particles in each three-level atom.

The linear terms appearing in the integrals of motion (33) and (34) are related to the energies of the atoms' levels, whereas the quadratic terms represent two kinds of interactions: (a) the terms with $\alpha \neq \beta$ include three different dipole–dipole interactions among the atoms, which are associated, respectively, to the transitions $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$ (see figure 1), (b) the terms with $\alpha = \beta$ are monopole interactions ($\mathbf{n}_{\alpha m} \mathbf{n}_{\alpha n}$).

We discuss now the meaning of the labels that characterize the $SU(3)$ irreps (the set $\left(\Lambda_m^1, \Lambda_m^2\right)$ and n_m, or $[k_{1m}, k_{2m}, k_{3m}]$). Without loss of generality, we can consider $k_{1m} = \Omega_m$

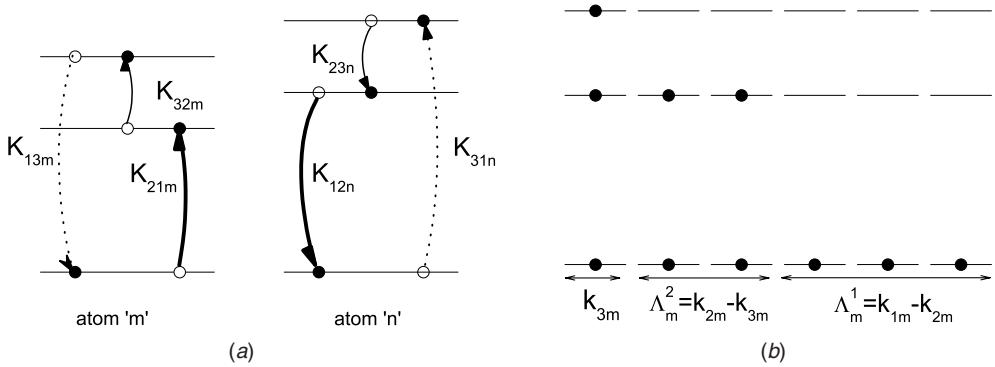

Figure 1. (a) Dipole–dipole transitions between atoms of type m and n. (b) Pictorial representation of a highest weight state for atoms of type m.

($\Omega_m - k_{1m}$ is the number of atoms without any particle in their levels, which are completely decoupled of the rest and do not contribute to the r_m eigenvalues). The atoms of each type can be classified according to the number of particle in their levels: (a) *unblocked atoms* are those with only a particle in their levels. These atoms have all the dipole transition allowed. (b) *Semi-blocked atoms* are those with two particles in their levels. The dipole transitions between the occupied levels are forbidden by Pauli blocking. (c) *Blocked atoms* are those with all their levels occupied. These atoms have all the dipole transitions forbidden, and interact with the others only by the monopole interaction. The Dynkin labels of the irreps determine the number of atoms of each type in the previous situations: $\Lambda_m^1 = k_{1m} - k_{2m} \leqslant \Omega_m$ indicates the number of type-m *unblocked atoms*, $\Lambda_m^2 = k_{2m} - k_{3m}$ is the number of *semi-blocked atoms* of type m, and the value k_{3m} indicates the number of *blocked atoms* of type m. A representation of a highest weight state is depicted in figure 1. The case where all the atoms are in a *unblocked* situation corresponds to $\Lambda_m^1 = \Omega_m$, $\Lambda_m^2 = 0$ and $k_{3m} = 0$. Note that integrals of motion (35) guarantee that the total populations (irrespective on the type of atom) in each level ($N_\alpha, \alpha = 1, 2, 3$) are conserved quantities. The number of parameters E_k and ω_l in the Richardson–Bethe equations (46) has a simple meaning: the number of E_k's (M_1) is equal to the number of atoms (of any type) with the first level unoccupied, whereas the number of ω_l's (M_2) is the overall number of unblocked and semi-blocked atoms with the third level occupied.

4.2. Particle–particle representation

Another possible realization of the $SU(3)$ algebra is

$$\begin{aligned}
&\mathbf{K}_{21m} = \mathbf{c}^\dagger_{\downarrow m}\mathbf{c}^\dagger_{0m} && \mathbf{K}_{12m} = \mathbf{c}_{0m}\mathbf{c}_{\downarrow m} && \mathbf{K}_{11m} = 1 - \mathbf{n}_{0m}\\
&\mathbf{K}_{31m} = \mathbf{c}^\dagger_{\uparrow m}\mathbf{c}^\dagger_{0m} && \mathbf{K}_{13m} = \mathbf{c}_{0m}\mathbf{c}_{\uparrow m} && \mathbf{K}_{22m} = \mathbf{n}_{\downarrow m}\\
&\mathbf{K}_{32m} = \mathbf{c}^\dagger_{\uparrow m}\mathbf{c}_{\downarrow m} && \mathbf{K}_{23m} = \mathbf{c}^\dagger_{\downarrow m}\mathbf{c}_{\uparrow m} && \mathbf{K}_{33m} = \mathbf{n}_{\uparrow m},
\end{aligned} \tag{48}$$

where $\mathbf{c}^\dagger_{\sigma m}$ and $\mathbf{c}_{\sigma m}$ are, respectively, fermionic creation and annihilation operators, and $\mathbf{n}_{\sigma m}$ are the number operators $\mathbf{c}^\dagger_{\sigma m}\mathbf{c}_{\sigma m}$. The two lowering operators $\mathbf{K}_{21}$ and $\mathbf{K}_{31}$ create a pair of particles and thus form a *spinorial* pair:

$$\mathbf{P}_m = \begin{pmatrix} \mathbf{P}^\dagger_{\uparrow m} \\ \mathbf{P}^\dagger_{\downarrow m} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{K}_{31m} \\ \mathbf{K}_{21m} \end{pmatrix}. \tag{49}$$

The other lowering operator ($\mathbf{K}_{32m} \equiv \mathbf{T}_{+m}$) is the ladder operator of a $SU_T(2)$ subalgebra of $SU(3)$:

$$\left[\mathbf{T}_{+m}, \mathbf{P}^{\dagger}_{\downarrow m}\right] = \mathbf{P}^{\dagger}_{\uparrow m}. \tag{50}$$

It is well known that the particle-particle representation of the RG models associated with the $SU(2)$ algebra describes the interaction between scalar pairs [12], whereas the $SO(5)$ version describes the vectorial pairing, i.e., the interaction between pairs forming a triplet [15]. The $SU(3)$ version presented here is associated with an interaction between pairs forming a doublet, we call it *spinorial* pairing or $T = 1/2$ pairing. As was done in [15] for the $SO(5)$ algebra, from a linear combination of the rational integrals of motion (33), the following pairing Hamiltonian for spinorial pairs results

$$\mathbf{H}_{T=1/2} = \sum_m z_m(\mathbf{n}_{\uparrow m} + \mathbf{n}_{\downarrow m}) + g \sum_{mn} \mathbf{P}^{\dagger}_m \cdot \mathbf{P}_n, \tag{51}$$

where the scalar product is $\sum_{\sigma=\uparrow\downarrow} \mathbf{P}^{\dagger}_{\sigma m}\mathbf{P}_{\sigma n}$, and we have fixed the parameters $\xi^1 = 1/g$ and $\xi^2 = 0$. In this particle–particle representation of the $SU(3)$ algebra, the Dynkin labels of the highest weight states are associated with the number of unpaired particles in each single particle level m and their transformation properties under the subgroup $SU_T(2)$. The number of variables E_k and ω_l is, respectively, the number of pairs and the number of $\uparrow$-particles in those pairs.

4.3. Tavis–Cummings models for three level systems and two bosons

In this section, we derive another integrable model, which consists of L copies of the $SU(3)$ algebra interacting with two different bosons. This model is derived from the trigonometric version of the previous section in two steps: first we take the bosonic mapping of one copy of the $SU(3)$ algebra (let us say the copy 0), and then we let the degeneracy Ω_0 of this bosonized copy go to infinite.

We assume the $SU(3)$ irrep $\left(\Lambda^1_0, \Lambda^2_0\right) = (\Omega_0, 0)$ for the copy to be bosonized. In this case the bosonic mapping for the $SU(3)$ algebra is [23]

$$\begin{array}{lll} \mathbf{K}_{11} \to \Omega_0 - \mathbf{a}^{\dagger}\mathbf{a} - \mathbf{b}^{\dagger}\mathbf{b} & \mathbf{K}_{22} \to \mathbf{a}^{\dagger}\mathbf{a} & \mathbf{K}_{33} \to \mathbf{b}^{\dagger}\mathbf{b} \\ \mathbf{K}_{31} \to \mathbf{b}^{\dagger}\sqrt{\Omega_0 - \mathbf{a}^{\dagger}\mathbf{a} - \mathbf{b}^{\dagger}\mathbf{b}} & \mathbf{K}_{32} \to \mathbf{b}^{\dagger}\mathbf{a} & \mathbf{K}_{21} \to \mathbf{a}^{\dagger}\sqrt{\Omega_0 - \mathbf{a}^{\dagger}\mathbf{a} - \mathbf{b}^{\dagger}\mathbf{b}}, \end{array} \tag{52}$$

where $\mathbf{a}^{\dagger}$ and $\mathbf{b}^{\dagger}$ are boson operators. This mapping is hermitic; therefore, the mapping of the operators $\mathbf{K}_{\alpha\beta}$ with $\beta > \alpha$ can be obtained by conjugating the previous ones. In the limit $\Omega_0 \to \infty$ the previous mapping is reduced to

$$\begin{array}{lll} \mathbf{K}_{11} \to \Omega_0 - \mathbf{a}^{\dagger}\mathbf{a} - \mathbf{b}^{\dagger}\mathbf{b} & \mathbf{K}_{22} \to \mathbf{a}^{\dagger}\mathbf{a} & \mathbf{K}_{33} \to \mathbf{b}^{\dagger}\mathbf{b} \\ \mathbf{K}_{31} \to \mathbf{i}\sqrt{\Omega_0}\mathbf{b}^{\dagger} & \mathbf{K}_{32} \to \mathbf{b}^{\dagger}\mathbf{a} & \mathbf{K}_{21} \to \mathbf{i}\sqrt{\Omega_0}\mathbf{a}^{\dagger}, \end{array} \tag{53}$$

where the imaginary factor $\mathbf{i}$ is introduced for future convenience. We consider now the trigonometric integrals of the RG model (34) for a system of $L + 1$ copies of the Lie algebra: $m = 0, 1, \ldots, L$. Then we bosonize the copy with the '0' label using (53). In order to avoid divergences in taking the limit $\Omega_0 \to \infty$, we define a new set of variables ϵ_m by using the freedom we have to choose the parameters z_m. These new variables are defined through

$$\cot(z_m - z_0) = \frac{\epsilon_m}{\sqrt{\Omega_0}}, \qquad m = 1, \ldots, L. \tag{54}$$

From this definition and the trigonometric identity ($\cot(A - B) = (\cot(A)\cot(B) + 1)/(\cot(B) - \cot(A))$, it is easy to show that in the limit $\Omega_0 \to \infty$

$$\cot(z_m - z_n) \to \frac{\sqrt{\Omega_0}}{\epsilon_n - \epsilon_m}. \tag{55}$$

By substituting these results in the trigonometric integrals $\mathbf{R}_0$ and $\mathbf{R}_m$ (34), we obtain

$$\hat{\mathcal{R}}_0 = -\zeta^1\mathbf{a}^\dagger\mathbf{a} - (\zeta^1+\zeta^2)\mathbf{b}^\dagger\mathbf{b} + \mathcal{C}_0\mathbf{1} + \sum_{n=1}^{L}(\mathbf{a}^\dagger\mathbf{K}_{12n} + \mathbf{a}\mathbf{K}_{21n} + \mathbf{b}\mathbf{K}_{31n} + \mathbf{b}^\dagger\mathbf{K}_{13n} + \epsilon_n\mathbf{K}_{11n})$$

$$\hat{\mathcal{R}}_m = -\zeta^1\mathbf{K}_{22m} - (\zeta^1+\zeta^2)\mathbf{K}_{33m} + \mathcal{C}_m\mathbf{1} + \sum_{n(n\neq m)}^{L}\frac{\sum_{\alpha\beta}\mathbf{K}_{\alpha\beta m}\mathbf{K}_{\beta\alpha n}}{\epsilon_m-\epsilon_n}$$
$$- (\mathbf{a}^\dagger\mathbf{K}_{12m} + \mathbf{a}\mathbf{K}_{21m} + \mathbf{b}\mathbf{K}_{31m} + \mathbf{b}^\dagger\mathbf{K}_{13m} + \epsilon_m\mathbf{K}_{11m}), \tag{56}$$

where we have rescaled the operators $\mathbf{R}_0$ and $\mathbf{R}_m$ ($\hat{\mathcal{R}}_0 \equiv \mathbf{R}_0/\sqrt{\Omega_0}$ and $\hat{\mathcal{R}}_m \equiv \mathbf{R}_m/\sqrt{\Omega_0}$), dropped the terms with inverse powers of $\sqrt{\Omega_0}$, and defined new variables:

$$\zeta^a = \frac{\xi^a}{\sqrt{\Omega_0}}. \tag{57}$$

The constants in the integrals of motion (56) are $\mathcal{C}_0 = -(1/3)\sum_l \epsilon_l n_l + (2\zeta^1+\zeta^2)(\Omega_0/3)$ and $\mathcal{C}_m = (\epsilon_m n_m)/3 + \sum_l' n_l/(\epsilon_l-\epsilon_m) + (2\zeta^1+\zeta^2)n_m/3$.

To obtain the eigenvalues of the previous operators, we take the eigenvalues (43), with $Z(z_j - z_i)$ and $Z(E_k - z_i)$ substituted by cotangent functions, and then the limit $\Omega_0 \to \infty$. Before performing this limit we introduce, as we did in (54), two new sets of variables associated with the parameters E_k and ω_k:

$$\cot(z_0 - E_k) = -\frac{x_k}{\sqrt{\Omega_0}}, \qquad \cot(z_0 - \omega_l) = -\frac{y_l}{\sqrt{\Omega_0}}. \tag{58}$$

The same trigonometric identity that yields (55) allows us to write the limit $\Omega_0 \to \infty$ of the following functions:

$$\cot(z_m - E_k) \to \frac{\sqrt{\Omega_0}}{x_k-\epsilon_m}, \qquad \cot(z_m - \omega_l) \to \frac{\sqrt{\Omega_0}}{y_l-\epsilon_m}, \qquad \cot(\omega_{l'} - \omega_l) \to \frac{\sqrt{\Omega_0}}{y_l-y_{l'}},$$
$$\cot(E_k - E_l) \to \frac{\sqrt{\Omega_0}}{x_l-x_k}, \qquad \cot(E_k - \omega_l) \to \frac{\sqrt{\Omega_0}}{y_l-x_k}, \tag{59}$$

where we have used the definition (54). With the limit values of the previous cotangent functions, the eigenvalues (43) become

$$\mathrm{r}_0 = \sum_{n=1}^{L} k_{1n}\epsilon_n + \mathcal{C}_0 - \sum_{k=1}^{M_1} x_k$$

$$\mathrm{r}_m = -\zeta^1 k_{2m} - (\zeta^1+\zeta^2)k_{3m} + \mathcal{C}_m + \sum_{n=1(n\neq m)}^{L}\frac{\sum_{\alpha=1}^{3}k_{\alpha m}k_{\alpha n}}{\epsilon_m-\epsilon_n} - k_{1m}\epsilon_m \tag{60}$$

$$+ (k_{1m}-k_{2m})\sum_{k=1}^{M_1}\frac{1}{x_k-\epsilon_m} + (k_{2m}-k_{3m})\sum_{l=1}^{M_2}\frac{1}{y_l-\epsilon_m}, \tag{61}$$

which are the eigenvalues of the operators (56). Note that the term $\mathcal{C}_0$, that diverges in the limit $\Omega_0 \to \infty$, appears explicitly both in the operator and its eigenvalue, then we can easily get rid of it. ϵ_m are free parameters and the variables x_k and y_l are solutions of the Richardson–Bethe equations in the limit $\Omega_0 \to \infty$. To derive them, we consider the Richardson–Bethe equations

in the trigonometric case, which can be read from equations (44) and (45) with Z substituted by cotangent functions. Then, using (59) we obtain

$$\sum_{k'(k'\neq k)}^{M_1} \frac{2}{x_{k'}-x_k} - \sum_{l=1}^{M_2} \frac{1}{y_l - x_k} - \sum_{m=1}^{L} \frac{k_{1m}-k_{2m}}{\epsilon_m - x_k} - x_k = -\zeta^1, \quad (k=1,\ldots,M_1)$$
$$-\sum_{k=1}^{M_1} \frac{1}{x_k - y_l} + \sum_{l'(l'\neq l)}^{M_2} \frac{2}{y_{l'}-y_l} - \sum_{m=1}^{L} \frac{k_{2m}-k_{3m}}{\epsilon_m - \omega_l} = -\zeta^2, \quad (l=1,\ldots,M_2). \tag{62}$$

Note that the resulting Richardson–Bethe equations are identical to those in the rational case except by the linear term in the first line. The integrals of motion (35) are now

$$\mathbf{N}_3 = \sum_{m=1}^{L} \mathbf{K}_{33m} + \mathbf{n}_b \qquad \mathbf{N}_2 = \sum_{m=1}^{L} \mathbf{K}_{22m} + \mathbf{n}_a \qquad \mathbf{N}_1 = \sum_{m=1}^{L} \mathbf{K}_{11m} + \Omega_0 - \mathbf{n}_a - \mathbf{n}_b. \tag{63}$$

The physical meaning of the Dynking labels $(\Lambda_m^1, \Lambda_m^2)$ and n_m is not modified by the introduction of the bosons, and it is the same already discussed above. The number of parameters x_k and y_l (M_1 and M_2) in the Richardson–Bethe equations (62) can be obtained from the general formula (46), by extending the sum to the bosonized copy, and noting that the Young labels associated with it are $[k_{10}, k_{20}, k_{30}] = [\Omega_0, 0, 0]$.

One of the simplest Hamiltonian that can be derived from the previous integrals of motion, is obtained by considering just one non-bosonized copy of the $SU(3)$ algebra. Therefore $L = 1$ and we have two integrals of motion: $\mathbf{R}_0$ and $\mathbf{R}_1$. By taking a linear combination of these integrals we arrive to

$$\begin{aligned}\mathbf{H} &= z_0(\hat{\mathcal{R}}_0 - \mathcal{C}_0\mathbf{1}) + z_1(\hat{\mathcal{R}}_1 - \mathcal{C}_1\mathbf{1}) \\ &= \omega_a\mathbf{n}_a + \omega_b\mathbf{n}_b + \eta_2\mathbf{K}_{22} + \eta_3\mathbf{K}_{33} + g(\mathbf{a}^\dagger\mathbf{K}_{12} + \mathbf{a}\mathbf{K}_{21} + \mathbf{b}\mathbf{K}_{31} + \mathbf{b}^\dagger\mathbf{K}_{13}),\end{aligned} \tag{64}$$

where $\mathbf{K}_{\alpha\beta}$ are the generators of the non-bosonized copy. The parameters of the Hamiltonian are related to the RG parameters by

$$\zeta^1 = \frac{\omega_a(\Delta_\eta - \Delta_\omega)}{g\Delta_\omega}, \qquad \zeta^2 = \frac{\Delta_\eta - \Delta_\omega}{g}, \qquad \epsilon_1 = \frac{\omega_a\Delta_\eta - \Delta_\omega\eta_2}{g\Delta_\omega}, \tag{65}$$

and the constants z_0 and z_1 are

$$z_0 = -g\frac{\Delta_\omega}{\Delta_\eta - \Delta_\omega}, \qquad z_1 = -g\frac{\Delta_\eta}{\Delta_\eta - \Delta_\omega}, \tag{66}$$

with $\Delta_\omega \equiv \omega_b - \omega_a$ and $\Delta_\eta \equiv \eta_3 - \eta_2$. The previous Hamiltonian describes the interaction between a set of Ω_1 identical three level atoms and two modes of the electric field in the so-called rotating-wave approximation [17], and is a generalized version of the Tavis–Cummings model for three level atoms. If we write the Hamiltonian (64) in the particle–particle representation, we obtain the interaction between a doublet of pairs and a doublet of bosons:

$$\mathbf{H} = \omega_+\mathbf{n}_+ + \omega_-\mathbf{n}_- + \epsilon_\uparrow\mathbf{n}_\uparrow + \epsilon_\downarrow\mathbf{n}_\downarrow + g(\mathbf{P}^\dagger\cdot\mathbf{b} + \mathbf{P}\cdot\mathbf{b}^\dagger), \tag{67}$$

where we have redefined the bosons $\mathbf{a}^\dagger \to \mathbf{b}_-^\dagger$, $\mathbf{b}^\dagger \to \mathbf{b}_+^\dagger$, and the scalar product, $\mathbf{P}^\dagger\cdot\mathbf{b}$, is $\mathbf{P}_\uparrow^\dagger\mathbf{b}_+ + \mathbf{P}_\downarrow^\dagger\mathbf{b}_-$. More complex Hamiltonians can be derived from the integrals (56), but we will not discuss them here.

5. Conclusions

In this contribution we introduced a Gaudin algebra valid for any simple Lie algebra in the rational, trigonometric and hyperbolic cases. With this algebra, we derived the complete set of quadratic RG integrals of motion, and found their respective eigenvalues. Focusing this formalism on the rank-2 $SU(3)$ algebra, we worked out in detail the RG models. For the rational case, expressions for the integrals of motion of degree three were obtained, and it was verified that their number is the necessary to match the number of integrals of motion with the number of degrees of freedom of the quantum model. Some physical applications of the $SU(3)$ models were discussed. The physical meaning of the variables in the RG formalism were discussed for two different representations of the $SU(3)$ algebra, namely, the particle–hole representation and the particle–particle one. By taking the trigonometric version of the models, we derived a new family of integrable models which are related to a generalization of the Tavis–Cummings model to three level atoms and two bosons. It was beyond the scope of this contribution to explore all the possible physical applications of the $SU(3)$ RG models; some examples were given and others are expected to come. We think that we have put the ground for more detailed studies, such as numerical studies of the solutions and comparison with approximative techniques in physical scenarios beyond the limits of traditional diagonalization methods. The formalism presented here can be easily applied to other Lie algebras. Some physically relevant examples are the $SU(4)$ and $SO(5)$ versions in the description of high temperature superconductors [24], the $SO(8)$ version for isovector–isoscalar pairing [25], and the $SU(n)$ versions which include generalized Tavis–Cummings models for n-levels atoms interacting with $n-1$ different bosonic modes. The formalism can likewise be extended to the elliptic RG models, where no linear term in the quadratic integrals of motion is allowed. Another interesting extension of the present contribution is to derive more general solutions to the Gaudin conditions (10), using the very well-studied solutions to the classical Yang–Baxter equations [26]. Finally, the higher rank RG models can be useful to shed some light in the non-completely well established definition of number of degrees of freedom in finite quantum systems. In this contribution, the definition of integrability coming from the Yang–Baxter equation was linked to that coming from the equality between the number of quantum degrees of freedom and the number of independent integrals of motion. This latter definition, initially supposed to be exclusively related to dynamical symmetric models, can be extended, as shown in this contribution, to the RG models, which are not necessarily dynamical symmetric.

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References

[1] Dukelsky J, Pittel S and Sierra G 2004 *Rev. Mod. Phys.* **76** 643
[2] Links J, Zhou H-Q, McKenzie R H and Gould M D 2003 *J. Phys. A: Math. Gen.* **36** R63–R104
[3] Richardson R W 1963 *Phys. Lett.* **3** 277
Richardson R W 1966 *Phys. Rev.* **141** 949

[4] Gaudin M 1976 *J. Phys. (Paris)* **37** 1087
[5] von Delft J and Poghossian R 2002 *Phys. Rev.* B **66** 134502
[6] Jurco B 1989 *J. Math. Phys.* **30** 1289
[7] Guan X-W, Foerster A, Links J and Zhou H-Q 2002 *Nucl. Phys.* B **642** 501
[8] Asorey M, Falceto F and Sierra G 2002 *Nucl. Phys.* B **622** 593
[9] Ushveridze A G 1994 *Quasi-Exactly Solvable Models in Quantum Mechanics* (Bristol: Institute of Physics Publishing) p 357
[10] Ortiz G, Somma R, Dukelsky J and Rombouts S 2005 *Nucl. Phys.* B **707** 421
[11] Amico L, Di Lorenzo A and Osterloch A 2001 *Phys. Rev. Lett.* **86** 5759
[12] Dukelsky J, Esebbag C and Schuck P 2001 *Phys. Rev. Lett.* **87** 066403
[13] Dukelsky J, Dussel G G, Esebbag C and Pittel S 2004 *Phys. Rev. Lett.* **93** 050403
[14] Links J, Zhou H-Q, Gould M D and McKenzie R H 2002 *J. Phys. A: Math. Gen.* **35** 6459
[15] Dukelsky J, Gueorguiev V, Van Isacker P, Dimitrova S, Errea B and Lerma H S 2006 *Phys. Rev. Lett.* **96** 072503
[16] Lerma H S, Errea B, Dukelsky J, Pittel S and Van Isacker P 2006 *Phys. Rev* C **74** 024314
[17] Yoo H I and Eberly J H 1985 *Phys. Rep.* **118** 239
[18] Chervov A, Rybnikov L and Talalaev D 2004 Rational Lax operators and their quantization *Preprint* hep-th/0404106
[19] Talalaev D 2006 *Funct. Anal. Appl.* **40** 73
[20] Zhang W M and Feng D H 1995 *Phys. Rep.* **252** 1
[21] Di Francesco P, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* (New York: Springer) p 489
[22] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley-Interscience)
[23] Klein A and Marshalek E R 1991 *Rev. Mod. Phys.* **63** 2
[24] Guidry M, Wu L-A, Sun Y and Wu C-L 2001 *Phys. Rev.* B **63** 134516
[25] Evans J A, Dussel G G, Maqueda E E and Perazzo P J 1981 *Nucl. Phys.* A **367** 77
[26] Belavin A A and Drinfel'd V G 1982 *Funct. Anal. Appl.* **16** 159